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Fresnel reflection from a cavity with net roundtrip gain

Tobias S. Mansuripur[1,a)] and Masud Mansuripur[2]
[1]*Department of Physics, Harvard University, Cambridge, Massachusetts 02138, USA*
[2]*College of Optical Sciences, The University of Arizona, Tucson, Arizona 85721, USA*

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A planewave incident on an active etalon with net roundtrip gain may be expected to diverge in field amplitude, yet applying the Fresnel formalism to Maxwell's equations admits a convergent solution. We describe this solution mathematically and provide additional insight by demonstrating the response of such a cavity to an incident beam of light. Cavities with net roundtrip gain have often been overlooked in the literature, and a clear understanding of their behavior yields insight to negative refraction in nonmagnetic media, a duality between loss and gain, amplified total internal reflection, and the negative-index lens. © *2014 AIP Publishing LLC*.
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The Fresnel coefficients govern the reflection and transmission of light for the simplest possible scenarios: at planar interfaces between homogeneous media. Despite this simplicity, some interesting solutions have been discovered only recently, such as (1) the amplification of evanescent waves in a passive, negative-index slab,[1–5] and (2) a duality between loss and gain leading to the localization of light in both cases.[6–10] In addition, controversy regarding the proper choice of the wavevector in active media has persisted in relation to the possibility of (3) negative refraction in nonmagnetic media,[11–15] as well as (4) single-surface amplified total internal reflection (TIR).[16–21] It turns out that all four of these cases share a common thread: the presence of a cavity whose roundtrip gain exceeds the loss. In this Letter, we explore in detail the Fresnel solution for such a cavity. We find that its peculiar properties help us to understand the four aforementioned phenomena and could also enable novel device functionalities.

To begin, we establish a convention that allows us to more clearly discuss the direction of energy flow. For the single-surface problem shown in Fig. 1(a), the incident wavevector in medium one is $\boldsymbol{k}_1^R = k_x\hat{\boldsymbol{x}} + k_{1z}^R\hat{\boldsymbol{z}}$, and the reflected wavevector is $\boldsymbol{k}_1^L = k_x\hat{\boldsymbol{x}} + k_{1z}^L\hat{\boldsymbol{z}}$, where $k_{1z}^L = -k_{1z}^R$. The superscript R (L) indicates that the wave carries energy to the right (left)—in other words that the time-averaged z-component of the Poynting vector is positive (negative). The real-valued component k_x, once determined by the incident wave, is the same for all wavevectors in the system. For the transmitted wavevector, the dispersion relation offers two choices for k_{2z}

$$k_{2z} = \pm\sqrt{(\omega/c)^2\mu_2\epsilon_2 - k_x^2}, \tag{1}$$

where ω is the angular frequency, c is the speed of light in vacuum, and μ_2 and ϵ_2 are the relative permeability and permittivity. It is universally agreed that the correct choice for k_{2z} in the single-surface problem is k_{2z}^R (i.e., that the transmitted energy flows away from the interface), irrespective of the material parameters or the nature of the incident wave, except possibly in the case of amplified TIR, for which there remains debate. Due to this controversy, let us postulate for now that k_{2z}^R is the correct choice in all cases, so that we can unambiguously define the single-surface Fresnel reflection and transmission coefficients

$$r_{\ell m} = \frac{\tilde{k}_{\ell z}^R - \tilde{k}_{mz}^R}{\tilde{k}_{\ell z}^R + \tilde{k}_{mz}^R}, \quad t_{\ell m} = \frac{2\tilde{k}_{\ell z}^R}{\tilde{k}_{\ell z}^R + \tilde{k}_{mz}^R}, \tag{2}$$

where we have generalized the result for incidence medium ℓ and transmission medium m. For s-polarization, we have defined $\tilde{k}_{nz} \equiv k_{nz}/\mu_n$, while for p-polarization $\tilde{k}_{nz} \equiv k_{nz}/\epsilon_n$. (In cases where both choices for k_{2z} result in no energy flow in the z-direction, such as for evanescent waves in a transparent medium, our prescription is to add a small amount of loss to the slab which will unambiguously distinguish k_{2z}^R and k_{2z}^L, then take the limit as the loss goes to zero.[22])

We now consider the case of light incident on a cavity, shown in Fig. 1(b). The total E-field resulting from an s-polarized incident wave in medium one with amplitude E_1^R is given by

$$E_y(x,z) = \begin{cases} E_1^R\exp(ik_xx + ik_{1z}^Rz) & \\ \quad +E_1^L\exp(ik_xx + ik_{1z}^Lz) & : z \le 0 \\ E_2^R\exp(ik_xx + ik_{2z}^Rz) & \\ \quad +E_2^L\exp(ik_xx + ik_{2z}^Lz) & : 0 \le z \le d \\ E_3^R\exp[ik_xx + ik_{3z}^R(z-d)] & : z \ge d, \end{cases} \tag{3}$$

where the time-dependence factor $\exp(-i\omega t)$ has been omitted. The most direct route to solve for the four unknown wave amplitudes is to enforce Maxwell's boundary conditions at $z = 0$ and $z = d$, which yields four equations that can be solved for the four unknowns. The resulting reflection coefficient from the slab can be expressed in terms of the single-surface Fresnel coefficients as

$$r \equiv \frac{E_1^L}{E_1^R} = \frac{r_{12} + r_{23}\exp(2ik_{2z}^Rd)}{1-\nu}, \tag{4}$$

where

[a)]Electronic mail: mansuripur@physics.harvard.edu

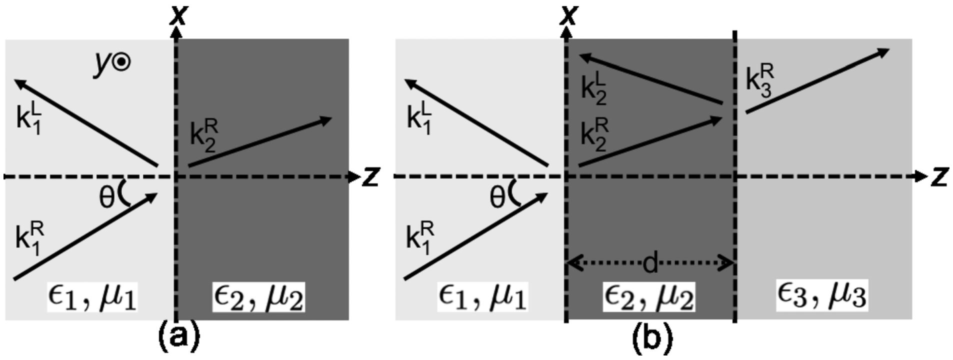

FIG. 1. Geometry of the (a) single-surface and (b) cavity problems. All media are infinite in the *x* and *y*-directions. The arrows denote the wavevectors present in each layer.

$$\nu = r_{21} r_{23} \exp(2ik_{2z}^R d) \tag{5}$$

is referred to as the roundtrip coefficient; the amplitude of a planewave circulating in the slab is multiplied by this factor after each roundtrip in the absence of any sources outside the slab. (Although we explicitly discuss *s*-polarized light, our conclusions as well as Eqs. (4) and (5) hold for both polarization states.) We emphasize that the reflection coefficient given by Eq. (4) is a valid solution to Maxwell's equations for any value of ν. The roundtrip coefficient ν has an important physical meaning, and intuitively one would expect three different regimes of behavior when the magnitude of ν is less than, equal to, or greater than one. The case where $|\nu| < 1$ governs passive slabs (in most but not all cases) and sufficiently weakly amplifying slabs. When $\nu = 1$ the slab behaves as a laser and emits light even in the absence of an incident wave, which manifests itself mathematically as an infinitely large reflection amplitude. The case where $|\nu| > 1$, however, is rarely openly acknowledged.[6,15,17]

Probably the reason for the neglect of the $|\nu| > 1$ steady-state solution is the seemingly intuitive assumption that the fields should diverge when there is net roundtrip gain. (See the supplemental material for a discussion of gain saturation and lasing.[22]) Unfortunately, this incorrect assumption seems to be reinforced by a second well-known solution method for the reflection coefficient that decomposes the reflected wave amplitude E_1^L into a sum over partial waves, yielding the reflection coefficient

$$r = r_{12} + t_{12} t_{21} r_{23} \exp(2ik_{2z}^R d) \sum_{m=0}^{\infty} \nu^m. \tag{6}$$

Heuristically, the first term r_{12} (hereinafter referred to as the "specular" partial wave) of Eq. (6) results from the single-surface reflection of the incident wave at the 1–2 interface, and the geometric series accounts for the contributions to the reflected wave following multiple roundtrips within the slab. When $|\nu| < 1$, the geometric series in Eq. (6) converges to $(1-\nu)^{-1}$, giving the same result as found by matching the boundary conditions in Eq. (4). When $|\nu| > 1$, however, the geometric series diverges and the partial wave method suggests that the reflection coefficient is infinite. Intuitively, this divergence seems reasonable, since we expect any light that couples into a slab with $|\nu| > 1$ to be amplified after each roundtrip, and therefore grow without bound. Nevertheless, Eq. (4) yields a finite reflection coefficient even when $|\nu| > 1$, so how can we reconcile these two very different solutions?

In fact, the usual heuristic interpretation of the partial wave picture does not tell the whole story, but with a slight modification the partial wave method *can* be used to find the $|\nu| > 1$ convergent solution. First, one can check that the reflection coefficient given by Eq. (4) is invariant under the transformation $k_{2z}^R \leftrightarrows k_{2z}^L$, provided $r_{23} \neq 0$. (This can be interpreted simply as a relabeling of the waves $E_2^R \leftrightarrows E_2^L$ in Eq. (3) that does not affect the final result.) Applying this same transformation to the partial wave sum of Eq. (6),[17] we can express the reflection coefficient as

$$r = r'_{12} + t'_{12} t'_{21} r'_{23} \exp(2ik_{2z}^L d) \sum_{m=0}^{\infty} \nu'^m, \tag{7}$$

where the prime indicates the substitution $k_{2z}^R \to k_{2z}^L$. Because the new roundtrip coefficient, $\nu' = r'_{21} r'_{23} \exp(2ik_{2z}^L d)$, is equal to ν^{-1}, in cases where $|\nu| > 1$ the primed partial wave sum of Eq. (7) will converge to the reflection coefficient of Eq. (4). (This duality between ν and ν^{-1} provides a simple mathematical explanation for the loss/gain duality observed by others.[6–10])

The physical implications of the substitution $k_{2z}^R \to k_{2z}^L$ in the partial wave sum can best be seen by examining the behavior of a "finite-diameter" beam of light incident obliquely on the slab. By numerically superposing a finite number of planewave solutions to Eq. (3)[22] with appropriate amplitudes and incidence angles in the range $27.47° < \theta < 32.53°$, we create a Gaussian (to within the sampling accuracy) beam incident on the slab at 30° with a full-width at half-maximum (FWHM) beam-diameter of 13.3 μm. All media are nonmagnetic, and we choose $\epsilon_1 = \epsilon_3 = 2.25$ and the slab to be an amplifying medium with $\epsilon_2 = 1 - 0.01i$. The free-space wavelength of the beam is $\lambda_o = 1\,\mu$m. We can examine the transition from $|\nu| < 1$ to $|\nu| > 1$ simply by varying d, since both $|r_{21}|$ and $|r_{23}|$ are less than one (and independent of d), whereas $|\exp(2ik_{2z}^R d)|$ (and hence ν) increases monotonically with d (because k_{2z}^R has a negative imaginary part). A plot of the field $E_y(x,z)$ at one instant of time is shown in Fig. 2(a) for $d = 19\,\mu$m, which was chosen so that $|\nu|$ is slightly less than one for all constituent planewaves of the beam ($0.46 < |\nu| < 0.99$). The arrows overlying the plot point in the direction of the time-averaged Poynting vector within their vicinity, indicating the direction of energy flow in the system, and the incident beam is uniquely identified by the white arrow. The beam behaves as we expect it to: the incident beam strikes the slab near ($x = 0$, $z = 0$), giving rise to a specularly reflected beam as

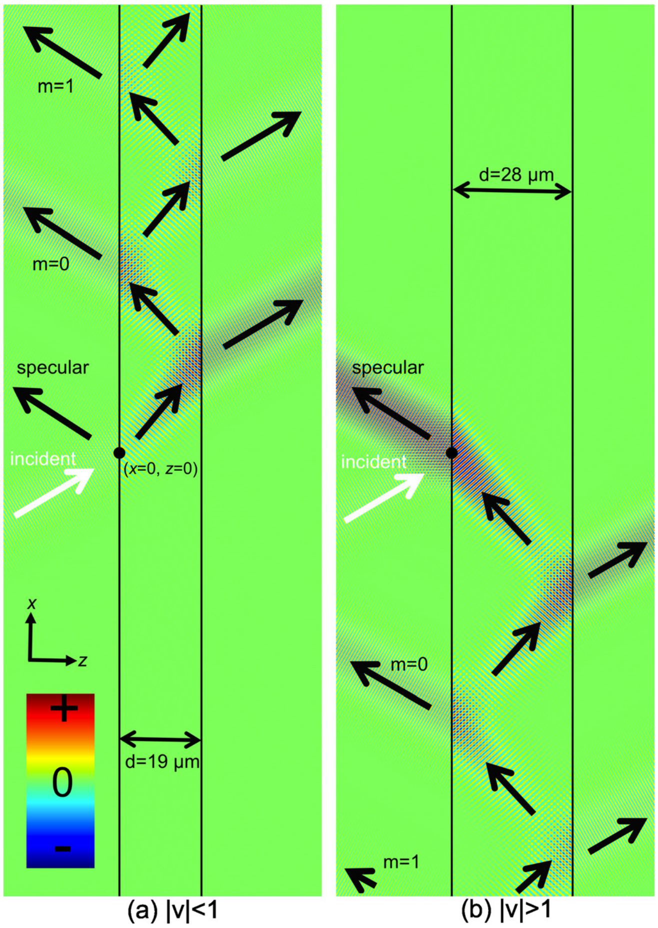

FIG. 2. Plots of the field $E_y(x,z)$ at one instant of time for a Gaussian beam (indicated with the white arrow) incident on an amplifying slab for which (a) $|\nu| < 1 (d = 19\,\mu\text{m})$ and (b) $|\nu| > 1$ $(d = 28\,\mu\text{m})$. The black dot indicates the origin of the coordinate system. For the same material parameters as (b), an incident pulse of light more vividly illustrates the peculiar behavior. (Multimedia view) [URL: http://dx.doi.org/10.1063/1.4869578.1]

well as a refracted beam that "zig-zags" up the slab, which in turn generates a reflected beam in medium one each time it strikes the 2-1 interface. (The field amplitude is plotted on a linear scale, and so the incident beam as well as the specularly reflected beam appear faint relative to the subsequently amplified portions of the beam.) Each of these reflected beams can intuitively be associated with a term of the partial wave expansion of Eq. (6)—either the specular term or the mth term of the geometric series.

In Fig. 2(b), all parameters are kept the same except the slab thickness is increased to $d = 28\,\mu\text{m}$, resulting in $|\nu| > 1$ for all constituent planewaves of the Gaussian beam $(1.01 < |\nu| < 2.58)$. Based solely on the plot of the field amplitude and not on the direction of energy flow indicated by the arrows, it may appear that the incident beam strikes the interface and negatively refracts in the slab, then zig-zags downwards in the $-\hat{x}$ direction, giving rise to many reflected beams in medium one (and transmitted beams in medium three) which emanate from points on the slab with $x < 0$. Such an explanation was offered for simulations similar to ours[14,15] to attempt to justify negative refraction in an active, nonmagnetic medium. However, by analyzing the Poynting vector we see that the energy in the beam zig-zags *up* the slab, so this phenomenon is distinct from negative refraction, despite the similarity in the positions of the reflected and transmitted beams. A video of a Gaussian pulse of light with a temporal FWHM of 50 fs and all other parameters identical to those of Fig. 2(b) more vividly illustrates that energy flows in the $+x$-direction. (The video plots the pulse intensity–not amplitude–on a logarithmic scale covering three decades.) We will refer to the field in the slab at $x < 0$ as the "pre-excitation," so-called because it occurs before the central lobe of the incident beam arrives at the slab. Each reflected beam in Fig. 2(b) can be associated either with the specular term r'_{12} or with the mth term of the primed partial wave expansion in Eq. (7).

The Fresnel solution for a slab with $|\nu| > 1$ is a steady-state harmonic solution, in the sense that if the field distribution presented in Fig. 2(b) exists at time $t_{o,}$ then as time is evolved forward the field at each point in space will vary harmonically with frequency ω. The intent of this Letter is not to investigate the causal evolution of the pre-excitation, beginning with the time the excitation source is turned on. We also recognize that the experimental verification of this potential phenomenon will be complicated by factors not included in the Fresnel formalism, such as spontaneous emission, that could lead to instabilities or self-lasing. Experimental work already done on the amplification of evanescent waves[16,23,24] (a regime for which $|\nu| > 1$), however, has not suffered from either of these problems. We note also that the Fresnel formalism, by implicitly beginning with the time-harmonic subset of Maxwell's equations, can only elucidate the non-divergent solutions to the full time-dependent equations. There can certainly exist divergent solutions, as demonstrated by finite-element simulations of a wave with a well-defined start-time incident normally on a slab with $|\nu| > 1$.[6]

Rather, we take the pre-excitation behavior demonstrated in Fig. 2(b) as the direct, logical, and inescapable consequence of the Fresnel formalism applied to Maxwell's equations for situations in which $|\nu| > 1$. This solution has surfaced in the literature,[1–21] sometimes knowingly but often not, but its peculiar properties have not been sufficiently appreciated. Our intent is merely to explore these properties and explain their relevance to some persistent controversies.

We observe that the specular reflection is given by r_{12} when $|\nu| < 1$ and r'_{12} when $|\nu| > 1$. Since $|r_{12}| < 1$ (in most cases of practical interest) and $r'_{12} = 1/r_{12}$, this means that $|r'_{12}| > 1$, and so the primed partial wave expansion mathematically predicts the amplification of the specularly reflected beam when $|\nu| > 1$. From Fig. 2(b), we see that this amplification occurs because the specular beam receives energy from the transmission of the pre-excited field through the 2-1 interface. Another noteworthy feature of this solution is that when $|\nu| \gg 1$ (achieved either by increasing the thickness or gain of the slab, or the incidence angle), the left-propagating wave amplitude E_{2L} becomes much larger than E_{2R}. This dominance of the left-propagating wave is of course a direct (although certainly peculiar) result of the multiple reflections of the pre-excitation at the front and back facets of the slab,[3] without which only the right-propagating wave E_{2R} would exist in medium two.

It turns out that TIR from an amplifying slab is well within the regime $|\nu| \gg 1$ (for any reasonable thickness d). As θ surpasses the critical angle for TIR, $|\nu|$ quickly becomes extremely large due to the negatively increasing $\mathrm{Im}(k_{2z}^R)$. (For the parameters used in Fig. 2(b), the critical angle is $\theta_c = 41.8°$. For $\theta = 41°$, $|\nu| = 9.34 \times 10^3$, and for $\theta = 42°$, $|\nu| = 1.40 \times 10^{15}$.) Therefore, only the left-propagating wave E_{2L} exists with any appreciable amplitude in the slab, and we emphasize again that this results directly from the multiple reflections of the pre-excitation at both slab facets. Some have argued that even if medium two is semi-infinite, above the critical angle the incident wave excites the wavevector k_{2z}^L in the transmission medium rather than the usual k_{2z}^R, resulting in the reflection coefficient r'_{12} and an accompanying amplified specular reflection.[16–19,21] It seems to us, however, that since the existing experimental results[16,24] can be explained by the slab picture without recourse to the single-surface problem, there is no need for a special TIR-exception to the postulate that k_{2z}^R is the transmitted wavevector in the single-surface problem.

In the case of the negative-index lens[1] (where media one and three are vacuum and medium two has $\epsilon_2 = \mu_2 = -1$), every incident evanescent wave ($k_x > \omega/c$), for both s and p-polarization, excites a lossless surface plasmon polariton mode[4] on the 1–2 and 2–3 interfaces, resulting in $r_{21} = r_{23} = \infty$ and hence $\nu = \infty$ (despite being a passive medium). Therefore, the convergent solution found by applying the Fresnel formalism to this problem[1,2] is the one described by the primed geometric series in Eq. (7) with $\nu' = 1/\nu = 0$. The result is that only the wavevector k_{2z}^L exists in the slab, which describes an evanescent wave that is amplified with increasing z, and therefore, enables the "perfect lensing" action. In this case, the reflection coefficient from the slab is given by $r = r'_{12} = 0$, and our argument indicates that this is the result of multiple reflections within the slab,[3] *not* because the incident evanescent wave is impedance-matched to the slab.[2,5] If loss is introduced to the slab, r_{21} and r_{23} become finite but the lens still works well if $|\nu| \gg 1$; however, even small losses lead to $|\nu| < 1$, causing the decaying wave k_{2z}^R to dominate the amplified wave k_{2z}^L, thereby spoiling the perfect lens.[4,5]

In conclusion, we have shown that the convergence of the Fresnel solution for a cavity with net roundtrip gain relies on the existence of the "pre-excited" field, which is a peculiar manifestation of the geometric partial wave series. By elucidating this counterintuitive phenomenon, we hope to have provided a useful alternative perspective for understanding amplified total internal reflection and the negative-index lens. We have also shown a positive-index slab with net roundtrip gain does not negatively refract–however, because the behavior mimics negative refraction insofar as the positions of the reflected and transmitted beams are concerned, the slab could substitute as a negative-index material in certain applications.

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